3 The Knebusch splitting tower

Notation 3.1. If not explicitly stated q, q' denote quadratic forms over the base field F.

Definition 3.2. Define recursively

$$F_0 \coloneqq F \;\; q_0 \coloneqq q_{\text{an}}$$
$$F_k \coloneqq F_{k-1}(q_{k-1}) \;\; q_k \coloneqq (q_{k-1})_{\text{an}} = (q_{F_k})_{\text{an}}$$

We call $i_k \coloneqq i_k(q) \coloneqq i(q_{F_k})$ the k-th Witt index. This number stabilizes after finitely many steps. Then first k s.t. q_{F_k} is split is called height of q and denoted by $h = h(q)$. Moreover, we call

$$F_0 \subseteq F_1 ... \subseteq F_h$$

the generic/Knebusch splitting tower.

Proposition 3.3.

$$\{i(q_k) | k = 0, 1, ..., n\} = \{i(q_K) | K/F \textit{ field extension}\}$$

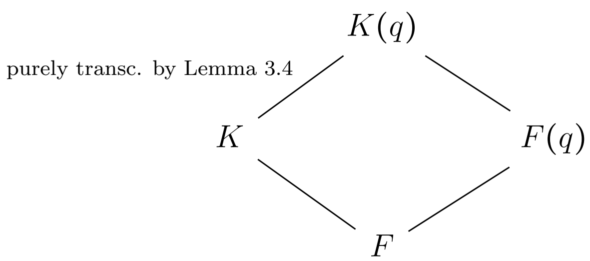
Lemma 3.4.

$$F(q)/F \textit{ is purely transcendental} \Leftrightarrow q \textit{ is isotropic}$$

Proof. "$\Rightarrow$": Recall that extending along purely transcendental extensions preserves anisotropic forms

"$\Leftarrow$": Hint: $\mathbb{H}$ is isometric to $(x, y) \mapsto xy$ and write x as rational function in terms of the other variables in the function field. □

Proof of Prop.3.3. Wlog q anisotropic. We just prove the statement by induction over $h(q)$: $h = 0$: Then q is hyperbolic and there is nothing to prove $h > 0$: Then consider $(q_{F(q)})_{\text{an}}$. This is a form of height one less, hence, by induction the statement holds. So it remains to show that for any field extension K/F making q_K anisotropic, that $i(q_K) \geq i_1(q)$ holds: But this is witnessed by the diagram



and the fact that if M/L is purely transcendental $i(f_M) = i(f)$ for any quadratic form f over L. □

4 The Separation Lemma of Hoffmann

Remark 4.1 ((Pfister) neighbor argument)**.** We will repeatedly use the following argument: Let $q \subseteq q'$. If $\dim q + i(q') > q'$, q' must also be isotropic as the underlying vector space of q and a totally isotropic subspace must intersect by dimension formula.

Remark 4.2 (Subform criterion)**.** Let g, f be quadratic forms over F. Then

$$g \subseteq f \Leftrightarrow i(f \perp -g) \geq \dim g$$

Proof. Hint: For "$\Leftarrow$" use the restriction on the Witt index to obtain a low-dimensional representative of $f - g$ in the Witt ring. Then use that equality in $W(F)$ and same dimension implies isometry. □

GOAL: Under which conditions does q stay anisotropic over $F(q)$.

Remark 4.3 (Motivation)**.** Define

$$q \preccurlyeq q' \Leftrightarrow q'_{F(q)} \text{ is isotropic}$$

if both $q \preccurlyeq q'$ as well as $q' \preccurlyeq q$ we write $q \approx q'$.

One easily shows that

$$q \preccurlyeq q' \Leftrightarrow \text{ There exists a rational map } Q \to Q'$$

[1]

Moreover, we have an equivalence

1. $q \approx q'$
2. Q and Q' are stably birational
3. The outer motivic summand of Q and Q' coincide

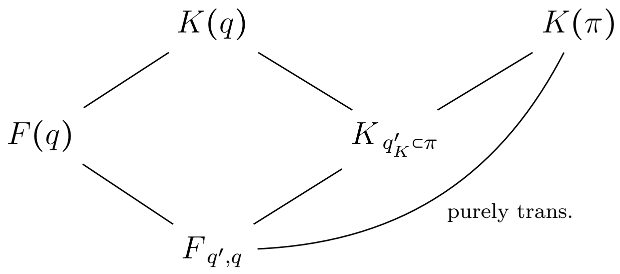
So the Hoffmann separation lemma gives restrictions not only for the algebraic theory but also in the geometric and motivic context.

Theorem 4.4 (Hoffmann Separation Lemma)**.** *If there exists $n \in \mathbb{N}$ s.t. $\dim q' \leq 2^n < \dim q$, then $q_{F(q')}$ is anisotropic*

Remark 4.5. Any strengthening of this result, cannot purely rely on an assertion on dimension: Indeed, Pfister neighbors q, q' of the same quadratic form will satisfy $q'_{F(q)}$ isotropic by a neighbor argument.

[1] One can even strengthen the statement of $\Rightarrow$ that there exists a dominant rational map $\mathbb{P}^n \times Q \to Q'$ by looking at function fields.

Remark 4.6. We will proceed in the following way to prove the separation lemma: Let n be maximal with the separating property.



First we will construct the extensions $L/K/F$, s.t.

1. L/F is purely transcendental and contains $K(\pi)$
2. There exists an anisotropic $(n+1)$-Pfister form π over K with $q'_K \subset \pi$

Finally we apply the following "Pfister neighbor argument": Suppose $q'_{F(q)}$ were isotropic. Then $\pi_{K(q)}$ is isotropic, hence, split. So by the main theorem about function fields $q_K \subset \pi$, i.e. Pfister neighbors. So q_L is isotropic contradicting L/F purely transcendental.

To start of let us tackle the first desired property. Fix $n \in \mathbb{N}$.

Lemma 4.7. *Let* $\pi = \langle\langle T_1, ..., T_{n+1} \rangle\rangle$ *over* $E = F(T_1, ..., T_n)$. *Then*

1. π *is anisotropic*
2. $L = E(\pi)/F$ *is purely transcendental*

Proof sketch. (1) follows from inductively applying the following result.

Claim. q_1, q_2 anisotropic quadratic forms over a field F. Then $q_1 \perp Tq_2$ is anisotropic over $F(T)$.

Proof. Exercise. Hint: Choose q_1, q_2 diagonal and kill denominators of a solution. Then compare coefficients. □

(2) just follows from the fact that the equation

$$0 = \pi(X_1, ..., X_{2^{n+1}}) = f(X_1, ..., X_{2^n}) + T_1 f(X_{2^n+1}, ..., X_{2^{n+1}})$$

, where $f = \langle\langle T_2, ..., T_{n+1} \rangle\rangle$, exhibits T_1 as a rational function in the other variables. □

Now we want to find an extension K/E such that K/E has the other desired propertes and $K(\pi)/E(\pi)$ is transcendental. For this we need the following criterion for a form to be a subform of a Pfister form

Lemma 4.8. *q, π annisotropic over F, π Pfister and* $\dim q < \dim \pi$. *Set* $\tilde{q} \coloneqq \pi \perp -q$. *TFAE:*

1. $\pi_{F(\tilde{q}_{an})}$ *is isotropic*

2. $q \leq \pi$

Proof. Note that the assumption on dimension implies that $\tilde{q}$ cannot be hyberpolic (neighbor argument). So condition (1) is always non-empty.

(2) implies (1) is clear. So assume $\pi_{\tilde{q}_{\mathrm{an}}}$ is hyperbolic. Then by the main theorem about function fields (and $1 \in D_F(\pi)$) we would like to deduce that $\pi \simeq \tilde{q}_{\mathrm{an}} \perp q'$ for some quadratic form q' over F:

For this observe that $\pi - \tilde{q}_{\mathrm{an}} = q$ in $W(F)$, hence, $\pi \perp -\tilde{q}_{\mathrm{an}}$ is isotropic over F by $\dim \pi > \dim q$. So $\tilde{q}_{\mathrm{an}}$ and π represent a common $a \in F^\times$.

Hence, in $W(F)$

$$\pi = \tilde{q}_{\mathrm{an}} + q' = \tilde{q} + q' = \pi - q + q'$$

As q is anisotropic, we obtain $q \simeq q'_{\mathrm{an}} \subseteq \pi$. □

Proof of Thm 4.4. With the notation of lemmas 4.7, 4.8

$$E = E_0 \subset E_1 \subset \ldots \subset E_h$$

be the Knebusch splitting tower associated to $\tilde{q} \coloneqq \pi \perp -q'_E$. Then we claim that the maximal i s.t. π_{E_i} is still anisotropic, satisfies

1. $i < h$, by neighbor argument as $\dim \pi > \dim \tilde{q}/2$

2. $(q'_{E_i})_{\mathrm{an}} \subset \pi_{E_i}$ as consequence of lemma 4.8

Observe that we know

1. $q'_{E(\pi)}$ is anisotropic by lemma 4.7

2. $\dim q' \leq \dim \pi/2$, by choice of n

from which we want to deduce that $E_i(\pi)/E(\pi)$ is purely transcendental:

Proving this resolves our remaining claims, namely:

- $K(\pi) = E_i(\pi)/E_{i-1}(\pi)/\ldots/E(\pi)/F$ is purely transcendental

- and therefore $q'_K = (q'_K)_{\mathrm{an}}$ as K is an intermediate field in $K(\pi)/F$.

Set $\tilde{q}_j \coloneqq (\tilde{q}_{E_j})_{\mathrm{an}}$. Observe that by induction it suffices to prove $(\tilde{q}_j)_{E_j(\pi)}$ is isotropic, whenever $\pi_{E_{j+1}}$ is anisotropic, which is equivalent to $j < i$.

Indeed, as we have $E_{j+1} = E_j(\tilde{q}_j)$ by definition and therefore $E_{j+1}(\pi) = E_j(\pi)(\tilde{q}_j)$.

To keep notation compact we may assume $j = 0$ (note that other than removing indices the assumptions and statement we want to prove do not change):

Suppose π_{F_1} is anisotropic. Aiming for contradiction, we assume $(\tilde{q}_{\mathrm{an}})_{E(\pi)}$ is anisotropic. So

$$(\tilde{q}_{\mathrm{an}})_{E(\pi)} = -q' \text{ in } W(E(\pi))$$

implies $(\tilde{q}_{\mathrm{an}})_{E(\pi)} = -q$ over $E(\pi)$ by assumption 2.. But this would imply

$$\dim \tilde{q}_{\mathrm{an}} = \dim q'$$

which would imply

$$\frac{\dim \pi}{2} \geq \dim q' = \dim \tilde{q} \geq \dim \pi - \dim q' \geq \frac{\dim \pi}{2}$$

where we used that if $\dim(\pi \perp -q')_{\mathrm{an}} < \dim \pi - \dim q'$, "$\leq$" would $\pi \simeq q' \perp \phi$ for some quadratic form ϕ over E by remark 4.2, "$<$" would imply that ϕ is isotropic - a contradiction. By the usual Witt ring argument this shows $\pi \simeq q' \perp \tilde{q}_{\mathrm{an}}$ over E, contradicting the fact that π_{F_1} is anisotropic, but $\tilde{q}_{F_1}$ is not. □